Mathematics Competition

Indiana University of Pennsylvania

2018

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.

2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.

3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.

4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.

5. Use of pencil, eraser, and scratch paper only are permitted.

6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. The Fibonacci numbers are a sequence of numbers with the pattern that any value in the sequence is found by adding the two previous values. The 1^{st} Fibonacci number is 1. The 2^{nd} Fibonacci number is also 1. The third is found by adding the prior two, so the 3^{rd} Fibonacci number is 2. The 4^{th} Fibonacci number is found by adding $1 + 2$ to get 3. We may continue in this way to get the sequence $1, 1, 2, 3, 5, \ldots$
   You may want to write out several terms of this sequence as we will use it again later in the contest on question #19 and question #46.

   This is the 55th annual IUP High School Mathematics Competition and 55 is one of the Fibonacci numbers. The true statement is:

   A. 55 is the 9^{th} Fibonacci number

   B. 55 is the 10^{th} Fibonacci number

   C. 55 is the 11^{th} Fibonacci number

   D. 55 is the 12^{th} Fibonacci number

   E. None of these

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2. If we solve for H in the equation $5 = \sqrt{kH}$, the solution is:

   A. $H = \dfrac{10}{k}$

   B. $H = \dfrac{\sqrt{5}}{k}$

   C. $H = \dfrac{5}{2k}$

   D. $H = \dfrac{k}{25}$

   E. $H = \dfrac{25}{k}$

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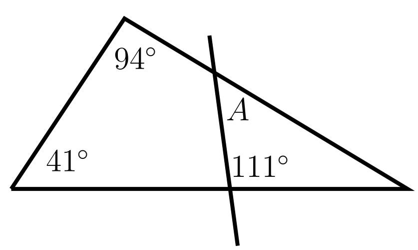
3. In the figure below, the measure of angle A is:



   A. 69°

   B. 24°

   C. 56°

   D. 28°

   E. 36°

---

4. The correct complete factorization of $a^2 - 2ab + b^2 - x^2 - 2xy - y^2$ is:

A. $(a+b)^2(x-y)^2$

B. $(a-b-x-y)(a-b+x+y)$

C. $(a-b-x+y)(a-b+x+y)$

D. $(a+b-x-y)(a+b-x+y)$

E. $(a-y)^2(b+y)^2$

---

5. The value of k that gives exactly one solution to the equation $3x^2 - 8x + k = 0$ is:

A. -24

B. $\dfrac{16}{3}$

C. $-\dfrac{12}{5}$

D. $\dfrac{3}{10}$

E. None of these

---

6. The solution for x in the equation $e^{10x+4} = 1$ is:

A. 0

B. $-\frac{3}{10}$

C. $\frac{1}{5}$

D. $\frac{2}{5}$

E. None of these

---

7. The set containing ALL solutions to $\dfrac{x}{x-5} - \dfrac{5}{x+5} = \dfrac{10x}{x^2-25}$ is:

A. $\{0, 4\}$

B. $\{2\}$

C. $\{5\}$

D. $\{2, 5\}$

E. None of these

---

8. If $3x - y - 12 = 0$, then the value of $\dfrac{8^x}{2^y}$ is:

A. 2^{24}

B. 2^{12}

C. 4^4

D. 8^2

E. None of these

---

9. The set containing all solutions to $2|3 - 2x| + 4 = 12$ is:

A. $\left\{-\dfrac{1}{2}\right\}$

B. $\left\{-\dfrac{7}{2}\right\}$

C. $\left\{\dfrac{1}{2}, -\dfrac{7}{2}\right\}$

D. $\left\{-\dfrac{1}{2}, \dfrac{7}{2}\right\}$

E. None of these

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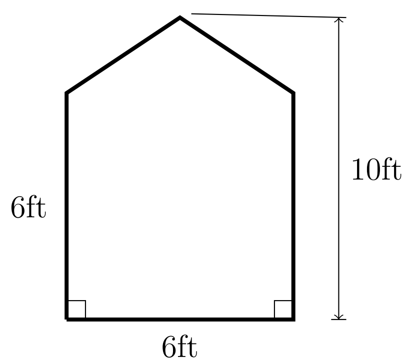
10. The area of the figure shown below is:



A. 34 ft

B. 34 ft^2

C. 48 ft

D. 48 ft^2

E. None of these

11. Let $f(x) = ax^2 + bx + c$ be the polynomial having the smallest degree that passes through the points $(1, 8)$, $(-1, 0)$, and $(0, 2)$. The value of abc is equal to:

A. 1

B. 4

C. 6

D. 8

E. None of these

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12. If $\sec(x) = -3$ and $\frac{\pi}{2} < x < \pi$, then the value of $\sin(x)\tan(x)$ is:

A. $\sqrt{8}$

B. $-\frac{8}{3}$

C. -1

D. $\frac{1}{3}$

E. None of these

---

13. Suppose $\frac{p}{q+r+s} = \frac{7}{2}$ and $\frac{p}{q+r} = \frac{2}{5}$. Then, the value of $\frac{s}{p}$ is:

A. $\frac{7}{6}$

B. $-\frac{11}{14}$

C. $-\frac{5}{7}$

D. $-\frac{31}{14}$

E. $\frac{14}{11}$

---

14. When the repeating decimal number $0.36363636\ldots = 0.\overline{36}$ is written into simplest reduced fractional form, the sum of the numerator and denominator is:

A. 137

B. 15

C. 9

D. 11

E. None of these

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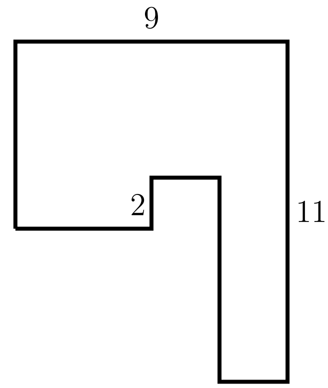
15. The perimeter of the figure given below is:



A. 40 units

B. 42 units

C. 44 units

D. 46 units

E. None of these

---

16. The average of m and 9 is x. The average of $2m$ and 15 is y. The average of 18 and $3m$ is z. Then, the average of x, y, and z is:

A. $m + 6$

B. $m + 7$

C. $2m + 14$

D. $3m + 21$

E. $6m + 42$

---

17. If the ratio of $2y - x + 6$ to $5y + 3x - 9$ is $-2/3$, then the ratio of y to x is:

A. $-\dfrac{3}{16}$

B. $\dfrac{4}{9}$

C. $-\dfrac{3}{8}$

D. $\dfrac{7}{5}$

E. None of these

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18. In the equation $2\log_b x = 2\log_b(1-a) + 2\log_b(1+a) - \log_b\left(\frac{1}{a} - a\right)^2$, with $0 < a < 1$ and $b > 0$, the solution for x is:

A. a

B. $-a$

C. $2a$

D. $\frac{1}{a}$

E. None of these

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19. Recall the definition of the Fibonacci numbers from question #1. Consider the following statements.

    I The 9$^{\text{th}}$ Fibonacci number is odd.

    II The 14$^{\text{th}}$ Fibonacci number is prime.

    III The 12$^{\text{th}}$ Fibonacci number is a perfect square.

    The statement(s) which must be true is:

    A. I only
    B. II only
    C. III only
    D. I , II, and III
    E. II and III only

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20. Consider the following four equations:

$$g = 3(3)^x\,; \quad h = 3^{2x}\,; \quad j = (3^3)^x\,; \quad k = 3^{2^x}\,.$$

    The equation(s) equal to 9^x would be:

    A. g and h only
    B. h and j only
    C. g and k only
    D. g, h, and k only
    E. None of these

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21. The equation $C = \frac{5}{9}(F - 32)$ shows how a temperature F, measured in degrees Fahrenheit, relates to a temperature C, measured in degrees Celsius. Consider the following statements.

    I A temperature increase of 1 degree Fahrenheit is equivalent to a temperature increase of $\frac{5}{9}$ degree Celsius.

    II A temperature increase of 1 degree Celsius is equivalent to a temperature increase of 1.8 degrees Fahrenheit.

    III A temperature increase of $\frac{5}{9}$ degree Fahrenheit is equivalent to a temperature increase of 1 degree Celsius.

    The statement(s) that must be true would be:

    A. I only
    B. II only
    C. III only
    D. I and II only
    E. II and III only

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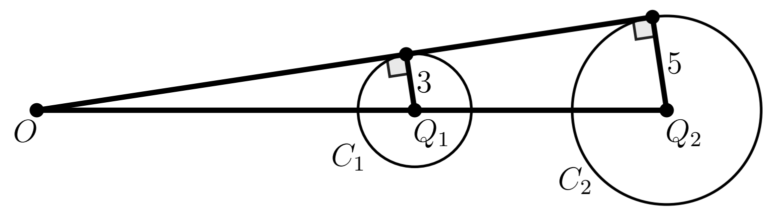
22. Given the coordinates $O(0,0)$ and $Q_1(20,0)$, the distance $|Q_1Q_2|$ is:



A. $\frac{40}{3}$

B. 20

C. $\frac{100}{3}$

D. 40

E. None of these

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23. The number of real-valued roots of $\sin^2(x) + \sin(x) - 2$ in the interval $[-13, 17]$ is:

A. 12

B. 9

C. 5

D. 2

E. 1

---

24. Let x be a real number. The unique real number y for which $xy = 1$ is called the **multiplicative inverse** of x. Now suppose that x is real number that satisfies the polynomial equation

$$5x^3 - 7x^2 + 4x - 3 = 0.$$

The multiplicative inverse of x is equal to:

A. $-x$

B. $\dfrac{1}{5x^3 - 7x^2 + 4x - 3}$

C. $3(5x^2 - 7x + 4)$

D. $\dfrac{1}{3}(5x^2 - 7x + 4)$

E. None of these

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25. If $f(x) = 2^x$, then the expression $f(x-1) + f(x+2)$ may be written as:

A. $3f(x)$

B. $f(x+1)$

C. $\dfrac{5}{2}f(x)$

D. $\dfrac{9}{2}f(x)$

E. $2f(x)$

---

26. A bacteria colony growing exponentially on a sandwich initially has a population of 10 bacterium. After 2 hours the colony has grown to a population of 6250 bacterium. The amount of time it will take for the bacteria colony to reach a size of 31250 is:

A. $\frac{5}{2}$ hours

B. 5 hours

C. $\frac{7}{2}$ hours

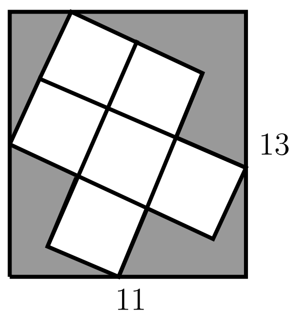
D. 10 hours

E. None of these

---

27. Determine the area of the shaded region in the figure below given the following:

- The interior 6 rectangles are congruent squares.
- Three of the squares placed end-to-end measure 11 units in length.



A. $\frac{187}{3}$ units^2

B. 89 units^2

C. 60.86 units^2

D. 65.24 units^2

E. None of these

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28. If two different sized hoses are used together to fill a pool, it takes 1 hour and 12 minutes. If the larger hose runs by itself, it takes one hour less than it does for the smaller hose to fill the pool by itself. The amount of time it takes the larger hose to fill the pool is:

A. 1 hour and 30 minutes

B. 2 hours

C. 2 hours and 20 minutes

D. 2 hours and 45 minutes

E. 3 hours

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29. Assuming x and y represent positive real numbers, solving for y in the equation $3x - \sqrt{xy} = \sqrt{x(x+3y)}$ yields:

A. $y = 3x$ or $y = -3x$

B. $y = 0$ or $y = 1$

C. $y = x$ or $y = 16x$

D. $y = x$ or $y = 9x$

E. $y = x^2$

---

30. If a cubic polynomial with real coefficients has a root of $3 - i$ and if the product of all of the roots is -5, then the real root of the polynomial is:

A. $3 - i$

B. 5

C. 1/2

D. -5

E. $-1/2$

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31. New cell phone numbers in a particular city are all of the form 432-555-#### where the last four digits may be any number, except no fives may be used. So, there are 6561 of these new phone numbers available. Of these new numbers, the total amount of different phone numbers with exactly four identical digits is:

A. 30

B. 72

C. 96

D. 102

E. None of these

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32. If $y = \cos(x)$ and $0 < x < \dfrac{\pi}{2}$, then the value of $\sin(2x)$ is:

A. $2y\sqrt{1-y^2}$

B. $\sqrt{1+y^2}$

C. $y\sqrt{y^2+1}$

D. $(2y^2+1)\sqrt{1-y^2}$

E. $\dfrac{1-y^2}{2}$

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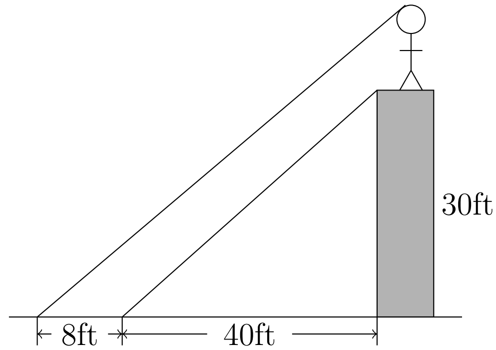
33. A building has a height of 30 feet and casts a shadow that is 40 feet long. A person is standing on the top of the building and casts a shadow that is 8 feet long. The height of the person is:



A. 5.5 ft

B. 6 ft

C. 6.25 ft

D. 6.5 ft

E. None of these

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34. The equation $x^3 - x^2 + 17x + 87 = 0$ has a solution of $x = -3$. Then the remaining solutions to the equation are:

A. $x = \dfrac{-4 \pm \sqrt{17}}{2}$

B. $x = 2 \pm 5i$

C. $x = \dfrac{2 \pm 7i}{4}$

D. $x = \dfrac{-3 \pm 2\sqrt{5}}{2}$

E. $x = -4 \pm i\sqrt{7}$

---

35. The set $\mathcal{B}_n$ represents the set of all binary sequences consisting of n digits. For instance,

$$x = 1001011101001$$

is a binary sequence contained in the set $\mathcal{B}_{13}$. In general, a sequence $x \in \mathcal{B}_n$ can be expressed as

$$x = d_1 d_2 \cdots d_n,$$

where $d_i = 0$ or $d_i = 1$ for $i = 1, 2, \ldots, n$. A sequence $x \in \mathcal{B}_n$ is called **even** if the sum of its digits is an even number and called **odd** if the sum of its digits is an odd number. The number of odd elements of $\mathcal{B}_{24}$ is equal to:

A. 24

B. 2^{12}

C. the number of even elements of $\mathcal{B}_{24}$

D. 2^{24}

E. None of these

---

36. Suppose the square root of p varies directly as the ratio of q to the square of r. We know $p = 16$ when $q = 24$ and $r = 2$. Then, when $p = 9$ and $q = 2$, the value is r is:

A. $r = 2/3$

B. $r = -1$

C. $r = 1$

D. $r = 2$

E. None of these

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37. The value of $\sin\left(\frac{\pi}{30}\right) + \sin\left(\frac{2\pi}{30}\right) + \sin\left(\frac{3\pi}{30}\right) + \ldots + \sin\left(\frac{60\pi}{30}\right)$ is:

A. 1

B. $\sqrt{2}$

C. -1

D. 0

E. $-\frac{\sqrt{2}}{2}$

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38. The exact value of the expression $\sum_{k=1}^{100} \log_k k^k$ is:

A. 50

B. 101

C. 5050

D. 10100

E. None of these

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39. The solution set for $2|x-3| - |x+4| \leq 8$ is:

A. $[-2, 18]$

B. $[-2, 1] \cup [4, 18]$

C. $[-4, -2] \cup [3, 18]$

D. $[-6, -2]$

E. None of these

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40. In 1949, Albert Einstein published a popular science article describing his childhood and his first proof. This proof relied on similarity to demonstrate the Pythagorean theorem. Below, the line segment (dashed line), drawn perpendicular to the hypoteneuse, yields three similar triangles T_a, T_b, and T_c. Let $A(T)$ represent the area of triangle T. Now, because the triangles are similar, the proportion p of the area of the triangle to area of the square is constant:

$$\frac{A(T_a)}{a^2} = \frac{A(T_b)}{b^2} = \frac{A(T_c)}{c^2} = p$$

For this problem, an express for p in terms of a, b, and c is:

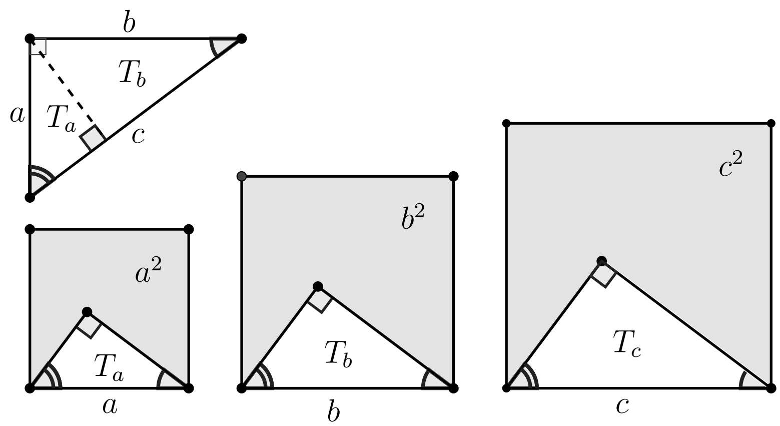
A. $p = \dfrac{ab}{c^2}$

B. $p = \dfrac{ab}{2c^2}$

C. $p = \dfrac{a+b}{c^2}$

D. $p = \dfrac{a+b}{2c^2}$

E. None of these



41. A function $f : \mathbb{R} \to \mathbb{R}$ is called a **linear transformation** provided that

$$f(ax + b) = af(x) + f(b)$$

for all real numbers a, b, and x. If f is a linear transformation, then the value of $f(0)$ is equal to:

A. $f(b)$

B. b

C. 1

D. 0

E. None of these

42. The graph $(x-3)^2 + (y-2)^2 = 9$ and the graph $y = x - 4$ meet at two points. The distance between the points is:

A. 3

B. $\sqrt{6}$

C. $2\sqrt{3}$

D. $3\sqrt{2}$

E. None of these

43. In a parabolic-type satellite dish, electromagnetic waves enter, hit the satellite dish, and are reflected to a single point P. A wave is said to be reflected if the incoming and outgoing waves both make an angle θ to the tangent line to the curve. Below, we assume the waves enter parallel to the x-axis and that P is located at the origin. By examining the figure below, we see that the slope m of the tangent line is $\tan(\theta)$. It can also be shown that $\tan(\pi - 2\theta) = -\frac{y}{x}$. Repeatedly use the trig identity

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

to express m in terms of x, y:

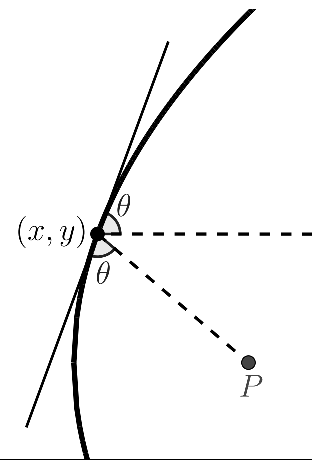
A. $m = \frac{-x+\sqrt{x^2+y^2}}{y}$

B. $m = \frac{-x-\sqrt{x^2+y^2}}{y}$

C. $m = \frac{\sqrt{x^2+y^2}}{y}$

D. $m = \frac{-\sqrt{x^2+y^2}}{y}$

E. None of these



44. The largest positive solution to $x^2 - 7x + \frac{1}{x^2} + 7\frac{1}{x} + 8 = 0$ is:

A. $1 + \sqrt{2}$

B. $\frac{1+\sqrt{15}}{2}$

C. $\frac{3+\sqrt{7}}{2}$

D. $\frac{5+\sqrt{29}}{2}$

E. $2 + \sqrt{17}$

45. The exact value of the expression $\frac{\log_3 \sqrt{243\sqrt{81\sqrt[3]{3}}}}{\log_2 \sqrt[4]{64} + \ln\left(e^{-10}\right)}$ is:

A. $43/51$

B. $-43/102$

C. $43/204$

D. $-86/102$

E. None of these

46. Recall the definition of the Fibonacci numbers from question #1. Since this is the 4th month of 2018, suppose we were to take the 2018^{th} Fibonacci number and divide by 4. We wish to determine the remainder of this division. However, instead of trying to determine such a large Fibonacci number, start at the beginning and write out several numbers in the sequence. Divide each by 4 and look for a pattern in the remainder of each division. From this, we may determine that the remainder when the 2018^{th} Fibonacci number is divided by 4 is:

A. 0

B. 1

C. 2

D. 3

E. None of these

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47. An equilateral triangle is circumscribed by a circle with radius r. Find r in terms of a, b, c:

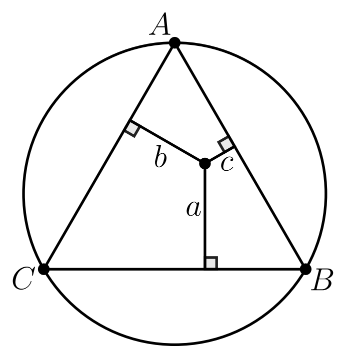
A. $r = \frac{1}{3}(ab + c)$

B. $r = \frac{1}{3}(ab - c)$

C. $r = \frac{2}{3}(abc)$

D. $r = \frac{1}{3}(a + b + c)$

E. None of these



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48. If $\tan^{-1}(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots$ for $-1 \le x \le 1$, then π can be expressed as:

A. $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dots$

B. $3\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dots\right)$

C. $4\left(-1 + \dfrac{1}{3} - \dfrac{1}{5} + \dfrac{1}{7} - \dots\right)$

D. $3\sqrt{3}\left(\dfrac{9}{5} - \dfrac{27}{7} + \dfrac{81}{9} - \dfrac{243}{11} + \dots\right)$

E. $2\sqrt{3}\left(1 - \dfrac{1}{9} + \dfrac{1}{45} - \dfrac{1}{189} + \dots\right)$

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49. For real numbers a and b, define $a \circ b$ according to the equation

$$a \circ b = a + b + ab.$$

Define a sequence of functions recursively according to the equations

$$\begin{aligned} & f_0(x) = x \circ 0 \\ \text{and} \quad & f_n(x) = x \circ f_{n-1}(x) \quad \text{for } n \geq 1. \end{aligned}$$

Now let $g(x) = f_{999}(x)$. The value of $g(-1)$ is equal to

A. -1

B. 0

C. 1

D. 1000

E. None of these

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50. Let circles C_1 and C_2 have centers $Q_1(x_1, 0)$ and $Q_2(x_2, 0)$ and radii r_1 and r_2, respectively. If $d = x_2 - x_1$, $r_*^2 = r_2^2 - r_1^2$, and $q_2 = \dfrac{d^2 + r_*^2}{2dr_2}$, then the area of the shaded region is

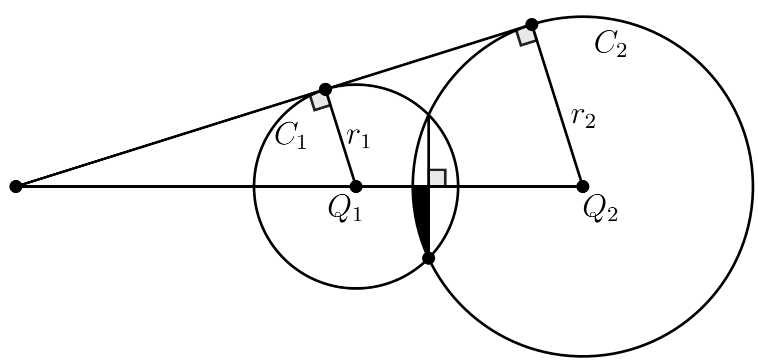
A. $r_2^2 \left(\cos^{-1}(q_2) - \sqrt{1 - q_2^2} \right)$

B. $2r_2^2 \left(\cos^{-1}(q_2) - \sqrt{1 - q_2^2} \right)$

C. $r_2^2 \left(\cos^{-1}(q_2) - q_2\sqrt{1 - q_2^2} \right)$

D. $\frac{r_2^2}{2} \left(\cos^{-1}(q_2) - q_2\sqrt{1 - q_2^2} \right)$

E. None of the above



Answer Key

1. B
2. E
3. B
4. B
5. B
6. E
7. E
8. B
9. D
10. D
11. E
12. B
13. D
14. B
15. C
16. B
17. A
18. A
19. C
20. E
21. D
22. A
23. C
24. D
25. D
26. A
27. A
28. B
29. C
30. E
31. D
32. A
33. B
34. B
35. C
36. A
37. D
38. C
39. A
40. B
41. D
42. D
43. A
44. D
45. B
46. B
47. E
48. E
49. A
50. D