Mathematics Competition

Indiana University of Pennsylvania

2014

Do not turn this page until directed by the proctor to do so.

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Answer Key

1. D
2. D
3. E
4. D
5. A
6. B
7. A
8. B
9. D
10. C
11. E
12. B
13. C
14. C
15. C
16. D
17. B
18. C
19. C
20. E
21. D
22. E
23. A
24. A
25. E
26. E
27. A
28. B
29. D
30. D
31. E
32. C
33. D
34. D
35. D
36. D
37. B
38. B
39. B
40. B
41. A
42. D
43. B
44. A
45. C
46. D
47. A
48. C
49. C
50. C

1. Line segment AB has endpoints $(2,-3)$ and $(-4,6)$. The coordinates of the midpoint of AB are:

   A. $(-2,3)$
   B. $(-1,3)$
   C. $\left(3,\frac{9}{2}\right)$
   D. $\left(-1,\frac{3}{2}\right)$
   E. None of these

---

2. The equation of the line perpendicular to $2x+3y=6$ and passing through the point $(8,3)$ is:

   A. $2x+3y=25$
   B. $3x+2y=30$
   C. $2x-3y=7$
   D. $3x-2y=18$
   E. None of these

---

3. Let $a=\frac{3}{5}$, $b=\frac{1}{3}$, and $c=\frac{5}{2}$. Then, ac^2-bc+a is equal to:

   A. $109/15$
   B. $121/60$
   C. $7/2$
   D. $13/4$
   E. None of these

---

4. For triangle $\triangle ABC$, you are given that $\cos(C)=1/2$, the length of AC is 7, and the length of BC is 3. The length of side AB is:

   A. 10
   B. $\sqrt{69}$
   C. 4
   D. $\sqrt{37}$
   E. $\pi/6$

---

5. The graph which could be used to find the solution to $y = -x + 2$ and $y = x^2$ is:

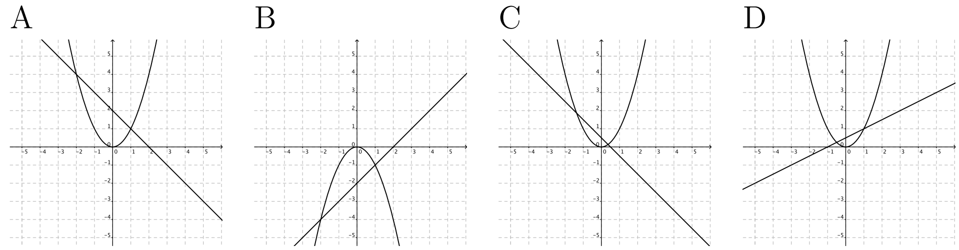
   A. Plot A.

   B. Plot B.

   C. Plot C.

   D. Plot D.

   E. None of these



---

6. The solution to the inequality $|2 - 4x| < 18$ is:

   A. $[-5, 5]$

   B. $(-4, 5)$

   C. $(-\infty, -4) \cup (5, \infty)$

   D. $(-\infty, -4] \cup [5, \infty)$

   E. $(-\infty, \infty)$

---

7. Let $\mathbb{R}$ denote the set of Real numbers, $\mathbb{N}$ denote the set of Natural numbers, and $\mathbb{Q}$ denote the set of Rational numbers. The true statement is:

   A. Every member of $\mathbb{N}$ is a member of $\mathbb{Q}$

   B. Every member of $\mathbb{R}$ is a member of $\mathbb{Q}$

   C. Every member of $\mathbb{Q}$ is a member of $\mathbb{N}$

   D. Every member of $\mathbb{R}$ is a member of $\mathbb{N}$

   E. All of the above statements are true

---

8. Let $f(x) = 4x + 5$, $g(x) = 3x - 1$, and $h(x) = 7 - 2x$. Define $P(x) = (f \circ g)(x)$ and $Q(x) = (g \circ h)(x)$. Then, $(Q \circ P)(1 - z)$ is:

   A. $72z - 147$

   B. $72z - 58$

   C. $72z + 50$

   D. $72z + 168$

   E. None of these

---

9. Given the figure below with altitude BD, median BF and BE as the bisector of angle ABC, the valid conclusion is:

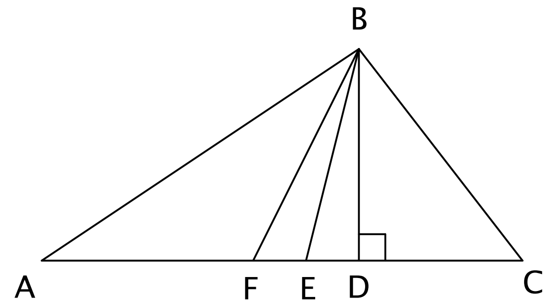
A. $\angle FAB = \angle ABF$

B. $\angle ABF = \angle CBD$

C. $CD = EA$

D. $CF = FA$

E. None of these



---

10. Given that $\tan\theta = 0.4$, the length of the side x is:

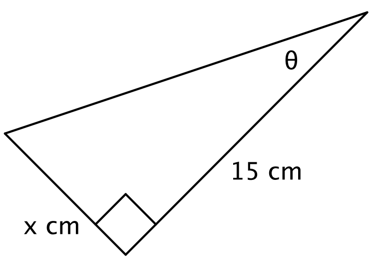
A. 37.5 cm

B. 12 cm

C. 6 cm

D. $\frac{5}{2}$ cm

E. None of these



---

11. Let n be a fixed positive integer. We say that two integers a and b are *congruent modulo* n, and write

$$a \equiv b \pmod{n},$$

if $a - b$ is divisible by n. The true statement is:

A. $18 \equiv 6 \pmod{7}$

B. $24 \equiv 14 \pmod{4}$

C. $12 \equiv 9 \pmod{7}$

D. $13 \equiv 9 \pmod{5}$

E. $14 \equiv 9 \pmod{5}$

---

12. Which of the following formulas describes the given graph:

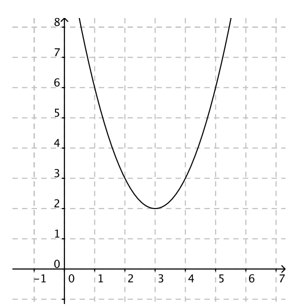
A. $y = -(x-3)^2 + 2$

B. $y = (x-3)^2 + 2$

C. $y = (x-3)^2 - 2$

D. $y = (x+3)^2 - 2$

E. $y = (x+3)^2 + 2$



---

13. Let (x, y) be a solution to the system of equations $2x - 3y = 8$ and $4x + 3y = -2$. Then, the product of x and y is equal to:

A. 1

B. 2

C. -2

D. -1

E. 4

---

14. A farmer has 120 feet of fencing. He wants to put a fence around three sides of a rectangular plot of land, with the side of a barn forming the fourth side. The maximum area he can enclose is:

A. 3600 square feet

B. 30 square feet

C. 1800 square feet

D. 60 square feet

E. None of these

---

15. An annulus is the solid region bounded by two concentric circles. The larger circle has radius R and the smaller circle has radius r. The area of the annulus lying within the first quadrant is:

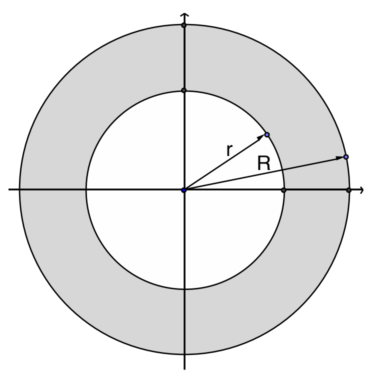
A. $\pi r^2 + R^2$

B. πR^2

C. $\frac{\pi}{4}(R^2 - r^2)$

D. $\pi(R^2 - r^2)$

E. None of these



---

16. Suppose $\ln(a) = \ln(a - 2) + 3$. The value of a is:

A. $\dfrac{3e^2}{1 + e^3}$

B. e^3

C. $\dfrac{2e^3}{1 + e^2}$

D. $\dfrac{-2e^3}{1 - e^3}$

E. $e^3 - 1$

---

17. Consider the system of equations $3u^3 - 3v = 6$ and $3u - v = 4$. A value of v in the ordered paired solutions (u, v) is:

A. -12

B. -10

C. 0

D. 4

E. 10

---

18. The interval that contains all solutions to $2x(x-2) = (x-2)(x+4)$ is:

A. $[-6, 0]$

B. $[-4, 3]$

C. $[0, 4]$

D. $[3, 6]$

E. $[5, 10]$

---

19. In the diagram, the circle has a radius of 5 and $CE = 2$. The length of BD is:

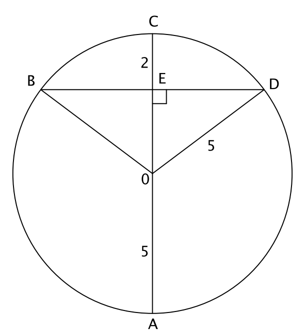
A. 12

B. 10

C. 8

D. 4

E. None of these



---

20. Suppose a and b are integers greater than 100 such that $a + b = 300$. A possible ratio of a to b is:

A. 9 to 1

B. 5 to 2

C. 5 to 3

D. 4 to 1

E. 3 to 2

---

21. The solution of $e^{2x} + e^x - 2 = 0$ is:

A. $\ln(-2)$
B. 1
C. $\ln(2)$
D. 0
E. None of these

---

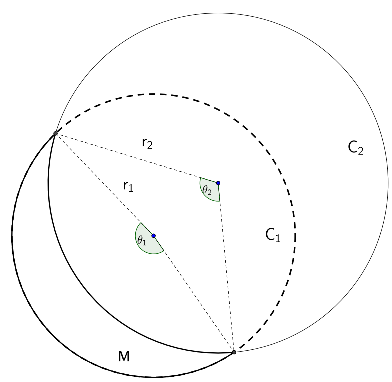
22. The vertex of the parabola $y = 3x^2 - 6x + 1$ is the point (h, k) where $h + 2k$ is:

A. -4
B. 19
C. 5
D. 3
E. -3

---

23. The crescent moon M is bounded by the edges of two circles C_1 and C_2 with radii r_1 and r_2, respectively. Circle C_1 coincides with M along an angle θ_1 (measured in radians) while circle C_2 coincides with M for an angle θ_2 (in radians). The perimeter of M is:

A. $r_1\theta_1 + r_2\theta_2$
B. $r_1\theta_1 - r_2\theta_2$
C. $r_2\theta_1 - r_2\theta_2$
D. $2r_1\theta_1 + 2r_2\theta_2$
E. None of these



---

24. A circle has a radius of $\log(a^2)$ and a circumference of $\log(b^4)$. Then $\log_a(b)$ is:

A. π
B. 2π
C. $\frac{\pi}{2}$
D. 4π
E. None of these

---

25. The domain for the rational function $f(x) = \frac{x-5}{3x^3-13x^2-10x}$ is:

A. $(-\infty, \infty)$

B. $(-\infty, -2/3] \cup [-2/3, 0] \cup [0, \infty)$

C. $(-\infty, -2/3] \cup [-2/3, 0] \cup [0, 5] \cup [5, \infty)$

D. $(-\infty, -2/3) \cup (-2/3, 0) \cup (0, \infty)$

E. $(-\infty, -2/3) \cup (-2/3, 0) \cup (0, 5) \cup (5, \infty)$

---

26. The number of positive factors of 25200 that are not divisible by 10 is:

A. 18

B. 30

C. 47

D. 48

E. None of these

---

27. In a race, athletes run three laps around an oval track formed by a rectangle and two semicircles as shown. The length of a radius of each semicircle is 12 meters. The length of the top of the rectangle is equal to twice the diameter of the semicircle. The total distance the athletes run during the race is:

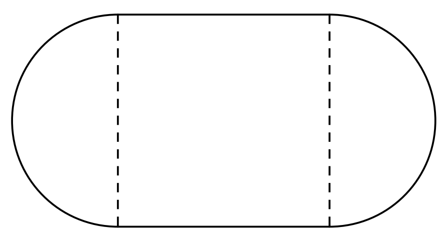
A. $288 + 72\pi$ meters

B. $96 + 24\pi$ meters

C. $288 + 144\pi$ meters

D. $144 + 72\pi$ meters

E. $96 + 12\pi$ meters



---

28. If $\log_2 5 = a$ and $\log_2 3 = b$, then $\log_2(0.9)$ in terms of a and b is:

A. $2a - b - 1$

B. $2b - a - 1$

C. $2b + a + 1$

D. $2a + b + 1$

E. None of these

---

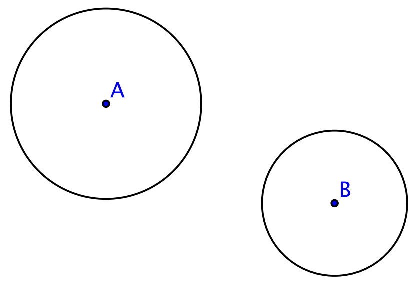
29. The solution set of $\frac{2}{x+1} > \frac{1}{x-2}$ is:

A. $(-\infty, -1) \cup (2, \infty)$
B. $(-1, 2)$
C. $(-\infty, -1) \cup (2, 5)$
D. $(-1, 2) \cup (5, \infty)$
E. None of these

---

30. The total number of lines of tangency that are common to both circle A and circle B is:

A. 1
B. 2
C. 3
D. 4
E. ∞



---

31. The diameter of a circle has the endpoints at $(-2, 3)$ and $(6, 9)$. The equation of the circle is:

A. $x^2 + y^2 + 15 = 0$
B. $x^2 + y^2 + 4x + 12y + 15 = 0$
C. $x^2 + y^2 - 4x + 9y - 45 = 0$
D. $x^2 - 4y + y^2 - 12y - 25 = 0$
E. None of these

---

32. The product of the ages of all the teenagers at Joyce's party last Saturday was 2,971,987,200. The number of 18 year olds who attended the party is:

A. 0
B. 1
C. 2
D. 3
E. 4

---

33. Let k be an odd integer. The number of integer solutions of the equation $x^2 + x - k = 0$ is:

A. 1

B. 2

C. 1 or 2 depending on the value of k

D. 0

E. None of these

---

34. Chuck and Dana agree to meet in Chicago for the weekend. Chuck travels 300 miles in the same time that Dana travels 264 miles. If Chuck's rate of travel is 6 mph more than Dana's, and they travel the same length of time, then Chuck's speed is:

A. 42

B. 38

C. 58

D. 50

E. 44

---

35. The number of subsets of $X = \{a, b, c, d, e, f, g, h, i, j\}$ not having the set $Y = \{a, b, c\}$ as a subset is:

A. 128

B. 512

C. 1536

D. 896

E. None of these

---

36. The smallest root of $f(x) = (\ln(x))^4 - 3(\ln(x))^2 + 2$ is:

A. $\sqrt{2}$

B. $\ln(2)$

C. e^{-1}

D. $e^{-\sqrt{2}}$

E. None of these

---

37. Suppose that, in the figure below, the shaded right-angled triangle is such that the length of AB is $\sqrt{3}$ and the length of DB is 1. The area of the triangular region that lies outside of the circle is:

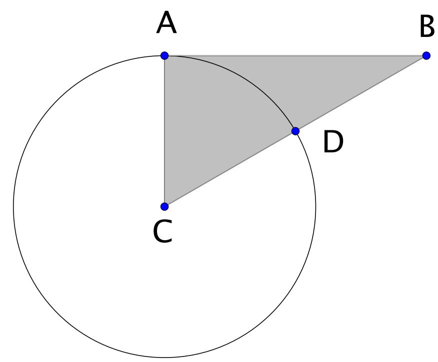
A. $\pi/6$

B. $(3\sqrt{3}-\pi)/6$

C. $(\pi-1)/2$

D. $\sqrt{3}/2-\pi$

E. None of these



---

38. Let n be the largest integer less than 10,000 that leaves a remainder of 1 when divided by any of the numbers 2, 3, 4, 5, 6, 7, or 8. The sum of the digits of n is:

A. 23

B. 16

C. 15

D. 13

E. 12

---

39. If $s = 1 + 3^1 + 3^2 + \cdots + 3^8 + 3^9$, then the value of s is:

A. 3^{10}

B. $\dfrac{3^{10}-1}{2}$

C. $\dfrac{3^9-1}{2}$

D. $\dfrac{1-3^8}{-2}$

E. None of these

---

40. If $\sin(\theta) = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\pi}{2} < \theta < \pi$, then $\sin^{-1}(\cos(\theta))$ is:

A. $120°$

B. $-30°$

C. $60°$

D. $-60°$

E. $150°$

---

41. Let $\log_2(x) = 2\log_2(3) - \dfrac{3}{2}\log_2(9)$. Then $3x + 8$ is:

A. 9

B. 10

C. 11

D. 17

E. $\dfrac{25}{2}$

---

42. Carla and Joe are painting. It takes Carla 24 minutes to paint a wall alone, and it takes 28 minutes for Joe to paint the same wall by himself. They both start painting the wall together, but Joe has to quit to run an errand while Carla finishes. Carla continues to work for exactly the same amount of time that both she and Joe had already worked together. How long from the start of the job did it take to paint the wall?

A. $12\frac{2}{7}$ minutes

B. $13\frac{2}{3}$ minutes

C. 14 minutes

D. $16\frac{4}{5}$ minutes

E. $17\frac{1}{2}$ minutes

---

43. In the circle shown with center O, the radius is 6. $QTSR$ is an inscribed square. Define w, x, y, and z to be the lengths of segments PQ, PT, PR, and PS respectively. The value of $w^2 + x^2 + y^2 + z^2$ is:

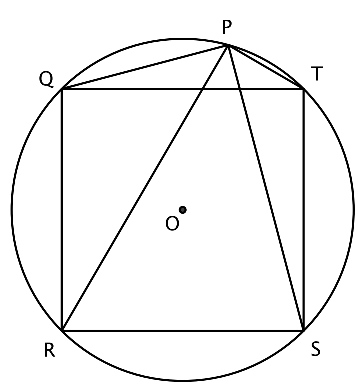
A. 432

B. 288

C. 36

D. 144

E. None of these



---

44. For real numbers x and y, define $x \circ y = x + y + xy$. Next, define a sequence of functions $\{f_1, f_2, f_3, \ldots\}$ recursively such that

$$f_1(x) = x$$

and for each natural number $n \geq 2$

$$f_n(x) = x \circ f_{n-1}(x).$$

Then the coefficient of x^{10} in the function $g(x) = 1 + f_{25}(x)$ is:

A. $3,268,760$

B. $360,360$

C. 10

D. 250

E. None of these

---

45. An exact value for $\sin\left(\frac{\pi}{16}\right)$ is:

A. $\frac{1-\sqrt{2}}{2}$

B. $\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{2}$

C. $\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$

D. $\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

E. $1+\frac{\sqrt{2}}{2}$

---

46. Evaluate

$$\log\left(\frac{1}{2}\right) + \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) + \cdots + \log\left(\frac{98}{99}\right) + \log\left(\frac{99}{100}\right)$$

A. 0

B. $-1/2$

C. -1

D. -2

E. None of these

---

47. If $r = x/y$ and $s = (x-y)/(x+y)$, then $4r/(1-r^2)$ is equivalent to:

A. $s - 1/s$

B. $s + 1/s$

C. $s/(s-1)$

D. $s^2 - s$

E. $1/(s+1)$

---

48. The product of all the solutions to $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 4$ on the interval $[-\pi, \pi]$ is:

A. $\dfrac{\pi^2}{18}$

B. $-\dfrac{\pi^2}{36}$

C. $-\dfrac{\pi^2}{9}$

D. $-\dfrac{25\pi^2}{36}$

E. None of these

---

49. Two circles C_1, C_2 each with radius r are centered at the origin O and P, respectively. Circle C_1 is fixed in the plane, but circle C_2 is rotating about C_1 in a counterclockwise manner while maintaining a point of tangency T. The x-coordinate of the point Q, which, before rotation through an angle θ, was the initial point of tangency is:

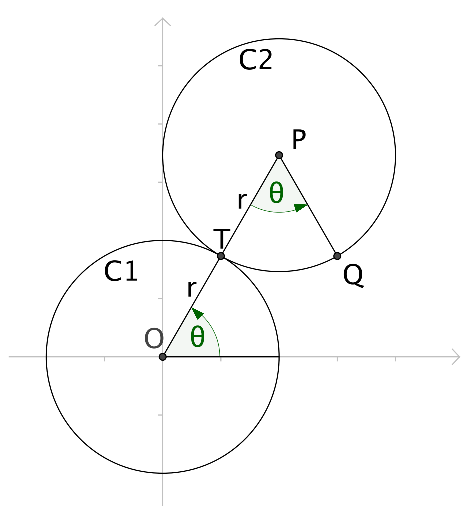
A. $2r\cos(2\theta) + r\cos(\pi + 2\theta)$

B. $2r\cos(\theta) + r\cos(\pi + \theta)$

C. $2r\cos(\theta) + r\cos(\pi + 2\theta)$

D. $r\cos(\theta) + 2r\cos(\pi + 2\theta)$

E. Impossible to determine



---

50. The multiplicative inverse of a 2×2 matrix A is the 2×2 matrix B such that$AB = BA = I_2$, where I_2 is the 2×2 identity matrix given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose that A is a 2×2 matrix such that

$$7A^2 - 3A + 4I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The multiplicative inverse of A is:

A. Does not exist

B. $\frac{1}{7}A^2 - \frac{1}{3}A + \frac{1}{4}I_2$

C. $\frac{3}{4}I_2 - \frac{7}{4}A$

D. $\frac{7}{4}A^2 - \frac{3}{4}A$

E. None of these