Mathematics Competition

Indiana University of Pennsylvania

2019

DIRECTIONS:

1. Please listen to the directions on how to complete the information needed on the answer sheet.
2. Indicate the most correct answer to each question on the answer sheet provided by blackening the 'bubble' which corresponds to the answer that you wish to select. Make your mark in such a way as to completely fill the space with a heavy black line. If you wish to change the answer, erase your first mark completely since more than one response to a problem will be counted wrong. Make no stray marks on the answer sheet as they may count against you.
3. If you are unable to solve a problem, leave the corresponding answer space blank on the answer sheet. You may return to it if you have time.
4. Avoid wild guessing since you are penalized for incorrect answers. If, however, you are able to eliminate one or more answers as being incorrect, the probability of guessing the correct answer is correspondingly increased. One-fourth of the number of wrong answers will be subtracted from the number of right answers. Therefore, guessing is discouraged. Due to the length of the test, you are not necessarily expected to finish it.
5. Use of pencil, eraser, and scratch paper only are permitted.
6. You will have 110 minutes of working time to do the 50 problems in the test. When time is called, put down your pencil and wait for additional instructions.

Do not turn this page until directed by the proctor to do so.

1. A *perfect number* is a positive integer that is equal to the sum of all of its divisors (other than itself). The smallest perfect number is 6, which is the sum of 1, 2, and 3. Another example of a perfect number is:

   A. 16

   B. 42

   C. 28

   D. 8

   E. None of these

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2. If $7x - 5y = 43$ and $3x + 2y = 6$, then $x - y$ is equal to:

   A. -7

   B. 1

   C. 31

   D. 55

   E. None of these

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3. In the figure below, the value of $a + b + c + d + e + f$ is:

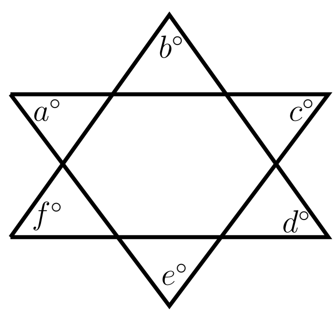
   A. 270

   B. 120

   C. 540

   D. 360

   E. 240



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4. Find the sum of the first six terms in the geometric sequence $2, -6, 18, -54, \ldots$

   A. -364

   B. 364

   C. -468

   D. 468

   E. None of these

---

5. A farmer used 220 feet of fencing to enclose a rectangular garden. If the area of the field is 2800 square feet, then the length of the longer side of the garden is:

A. 50 feet

B. 60 feet

C. 70 feet

D. 80 feet

E. 100 feet

---

6. The complex fraction $\dfrac{x^{-1}+y^{-1}}{\frac{3}{xy}}$ simplifies to:

A. $\dfrac{2xy}{3x+3y}$

B. $\dfrac{-x^2y-xy^2}{3}$

C. $\dfrac{xy}{3x+3y}$

D. $\dfrac{-xy}{x^2y+xy^2+3}$

E. $\dfrac{x+y}{3}$

---

7. The value $\sin^2(56°)+\sin^2(34°)$ is equal to:

A. 0

B. $1+\cos^2(34°)$

C. 1

D. $1+\sin^2(34°)$

E. $\tan^2(56°)$

---

8. The expression $\log_5\sqrt[3]{\dfrac{25}{a^2+b^2}}$ is equal to:

A. $\frac{1}{3}[2-\log_5(a^2+b^2)]$

B. $\frac{1}{3}\log_5(25)-\log_5(a^2+b^2)$

C. $3[2-\log_5(a^2+b^2)]$

D. $3\log_5(25)-\log_5(a^2+b^2)$

E. $\frac{1}{3}[2-\log_5(a^2)+\log_5(b^2)]$

---

9. The rectangle below is made up of 12 congruent squares. The area of the rectangle is 432 cm^2. The perimeter of the rectangle is:

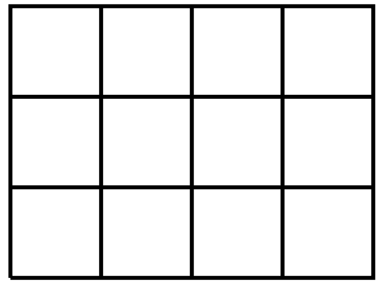
   A. 48 cm

   B. 84 cm

   C. 60 cm

   D. 72 cm

   E. 72 cm^2



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10. The function $f(x)$ is defined to be

$$f(x) = \begin{cases} x^2 - 2, & x < 0 \\ 1 - 2x, & x \geq 0\ . \end{cases}$$

The value of $f(0)$ is:

A. 0

B. -2

C. 1

D. Both -2 and 1

E. None of these

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11. The interval of the x-axis that contains all of the solutions to $6x^3 - 9x^2 - 10x + 15 = 0$ is:

A. $(1, 2)$

B. $(-4/3, 4/3)$

C. $(-2, 1)$

D. $(-1, 2)$

E. None of these

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12. If $f(x) = 3x - 2$, then $xf(x - 2) - 3f(x)$ is equal to:

A. $3x^2 - 11x + 4$

B. $3x^2 - 17x - 6$

C. $3x^2 - 11x - 8$

D. $3x^2 - 17x + 6$

E. $3x^2 - 11x - 6$

---

13. If the sum of five consecutive positive integers is A, then the sum of the next five consecutive integers in terms of A is:

A. $A + 1$

B. $A + 5$

C. $A + 25$

D. $2A$

E. $5A$

---

14. Recall the definition of a perfect number from question #1. We may also define a *semiperfect number* to be a positive integer that is equal to the sum of some of its divisors (other than itself). An example of a semiperfect number is 12, which is the sum of 2, 4 and 6. We did not use divisors 1 and 3, so 12 is not perfect. Another example of a semiperfect number is:

A. 10

B. 16

C. 32

D. 56

E. None of these

---

15. The value of $\sin(75^\circ)$ is equal to:

A. $\dfrac{1-\sqrt{2}}{\sqrt{3}}$

B. $\dfrac{1+\sqrt{3}}{\sqrt{8}}$

C. $\dfrac{\sqrt{3}}{2}$

D. $\dfrac{1+\sqrt{3}}{\sqrt{2}}$

E. $\dfrac{1-\sqrt{3}}{\sqrt{2}}$

---

16. $AB+CD = AAA$, where AB and CD are two-digit numbers and AAA is a three-digit number. A, B, C, and D are distinct positive integers. The value of C is:

A. 1

B. 3

C. 7

D. 9

E. None of these

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17. The inner square is constructed using the midpoints of the outer square. Let P be the midpoint of $\overline{AB}$. The angle α is:

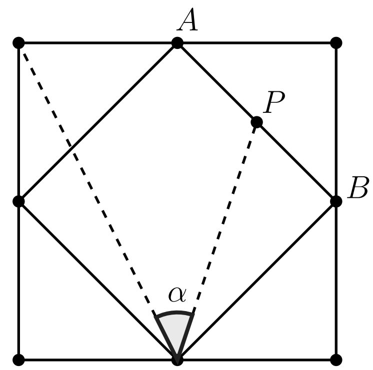
A. 22.5°

B. 30°

C. 45°

D. 60°

E. None of these



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18. One solution of $\dfrac{33}{2x^2+x-6} - \dfrac{x-2}{3x+6} = 2$ is $x = -3$. The other solution is:

A. $\dfrac{43}{14}$

B. $\dfrac{47}{14}$

C. $\dfrac{53}{26}$

D. $\dfrac{73}{14}$

E. $\dfrac{165}{26}$

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19. Let $m > 1$ be a positive integer and write $k = \log_{\frac{1}{m}}(y_k)$. Then, the sum $\displaystyle\sum_{i=1}^{\infty} y_i$ is equal to:

A. m

B. $\dfrac{1}{m}$

C. $m-1$

D. $\dfrac{1}{m-1}$

E. None of these

---

20. The intersection of two sets A and B, denoted by $A \cap B$, consists of all objects that are in both sets. In set-builder notation,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

For each natural number n, define an interval of real numbers according to the equation,

$$A_n = \left(-3 - \frac{1}{n},\, 7 + \frac{1}{n}\right).$$

Let $X = A_1 \cap A_2 \cap A_3 \cap \cdots$. In other words, X consists of all real numbers that are in each of the sets A_n for $n = 1, 2, \ldots$. The set X can also be expressed as:

A. $(-3, 7)$
B. $(-3, 7]$
C. $[-3, 7)$
D. $\emptyset$
E. None of these

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21. The cost of gas rose by 2 cents per gallon from last week to this week. Last week Steve bought 20 gallons of gas at the old price. This week he bought 10 gallons at the new price. Altogether, Steve spent \$66.20 on gas. The old price for one gallon of gas is:

A. \$1.52
B. \$1.54
C. \$2.18
D. \$2.20
E. \$2.22

---

22. The sum of all solutions of $(x^2 + 6x + 2)^2 - 3(x^2 + 6x + 2) = 54$ is:

A. -12
B. 3
C. 16
D. -42
E. None of these

---

23. The option that contains *all* solutions to $e^{2x} - 5e^x - 14 = 0$ is:

A. $x = \ln 7$
B. $x = -2$ and $x = 7$
C. $x = 7$
D. $x = \ln(-2)$ and $x = \ln 7$
E. None of these

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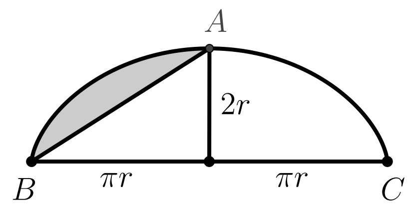
24. The total area under the cycloid BAC and above the base is $3\pi r^2$. The area of the shaded region is:



A. $\frac{1}{2}\pi r^2$

B. πr^2

C. $\frac{3}{4}\pi r^2$

D. $\sqrt{2}\pi r^2$

E. None of these

---

25. A function p whose domain and range is the set of real numbers $\mathbb{R}$ is called a projection provided that for every x_1, x_2 in $\mathbb{R}$ and for all constants a, b in $\mathbb{R}$,

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2) \quad \text{and}$$
$$p(p(x_1)) = p(x_1).$$

Suppose that x_0 is a real number in the domain of p. The value of $p(x_0 - p(x_0))$ is always equal to:

A. x_0

B. $p(x_0)$

C. $x_0 - p(x_0)$

D. 1

E. None of these

---

26. The expression $\sin^2(2\alpha) + (2\cos^2\alpha - 1)\left(1 - 2\sin^2\alpha\right)$ may be simplified to:

A. 1

B. π

C. $\alpha/2$

D. 2α

E. None of these

---

27. The solution set of $5^{\frac{x^3+4x^2+3x}{x+3}} = 15625$ is:

A. $\{-3\}$

B. $\{-2\}$

C. $\{2\}$

D. $\{-3, 2\}$

E. None of these

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28. A right triangle is shown below. The value of $x + z$ is:

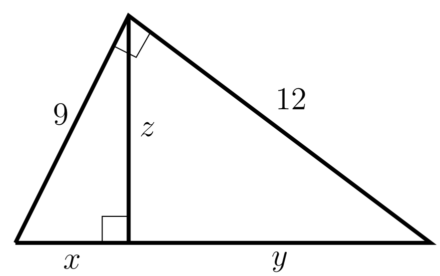
A. 9

B. 9.15

C. 10.4

D. 12.6

E. 15



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29. A poll was taken, and it was found that 12 students went to the movies on Friday, 10 students went to the movies on Thursday, 4 went on both days, and 7 did not go on either day. The percent of all students polled who went to the movies only on Thursday is:

A. 40%

B. 6%

C. 10%

D. 24%

E. 30%

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30. The second smallest number in the list 25^{60}, 16^{30}, 32^{24}, 27^{40}, 4^{120} is:

A. 25^{60}

B. 16^{30}

C. 32^{24}

D. 27^{40}

E. 4^{120}

---

31. When 4 is added to two numbers, the ratio is $5 : 6$. When 4 is subtracted from the same two numbers, the ratio is $1 : 2$. If the two numbers are x and y and $x < y$, then the value of $2x + y$ is:

A. 20

B. 22

C. 24

D. 26

E. 28

---

32. At what point or points, if any, does the graph of the given function cross its horizontal asymptote?

$$f(x) = \frac{3x^2 + 4x + 1}{x^2 + 3}\ .$$

A. $(3, 2)$

B. $(2, 3)$

C. $(0, -1)$

D. The function does not have a horizontal asymptote

E. The graph of the function does not cross the horizontal asymptote

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33. The number 730400 has 2 trailing zeroes. Also, the symbol ! means factorial, or the product of an integer and all the positive integers below it (e.g. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$). The number of trailing zeros in the number 2019! is:

A. 195

B. 380

C. 403

D. 502

E. 614

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34. A commercial airplane is detected 1000 miles north, as it is flying north with a speed of 700 miles per hour. Thirty minutes later, an interceptor plane flying with a speed of 800 miles per hour is deployed. The amount of time it will take the interceptor plane to reach the other plane is:

A. 12.5 hours

B. 13.0 hours

C. 13.5 hours

D. 14.0 hours

E. 14.5 hours

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35. The area of the given shape is:

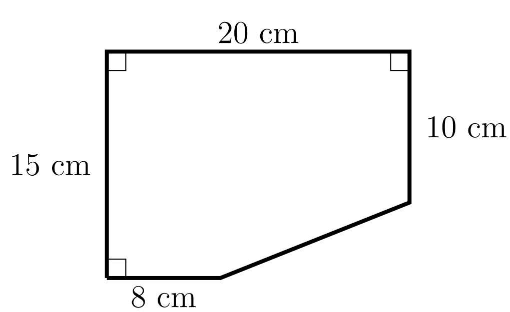
A. 240 cm^2

B. 250 cm^2

C. 260 cm^2

D. 270 cm^2

E. 280 cm^2



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36. For a natural number k, a binary string of length k is an expression of the form

$$B = b_1 b_2 \cdots b_k,$$

where either $b_i = 0$ or $b_i = 1$ for $i = 1, 2, \ldots, k$. The weight of a binary string B, denoted by $w(B)$, is defined according to the equation

$$w(B) = b_1 + b_2 + \cdots + b_k.$$

The binary string B is **even** provided that $w(B)$ is an even integer and the binary string B is **odd** provided that $w(B)$ is an odd integer. The number of binary strings of length k having even weight is equal to:

A. 2^k

B. $2^{(1/2)k}$

C. 2^{k-1}

D. $\frac{1}{2}k$

E. None of these

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37. The vertices of the inscribed square bisect the sides of the second square. The ratio of the area of the outer square to the area of the inscribed square is:

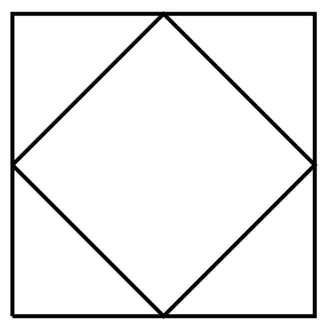
A. $2 : 1$

B. $1 : 2$

C. $\sqrt{2} : 1$

D. $1 : \sqrt{2}$

E. $4 : 1$



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38. Assume the area of ΔPQR is 1, and further assume that the ratio of lengths is $|PT|/|PQ| = k$. The area of the larger triangle is

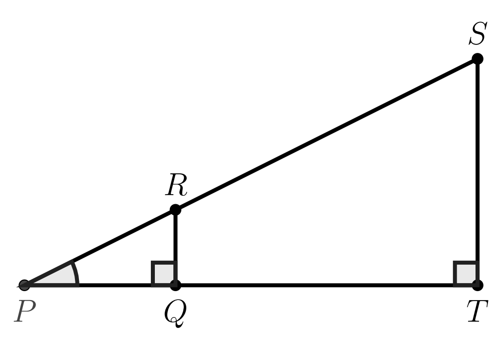
A. $\frac{1}{2}k$

B. k

C. $\frac{1}{2}k^2$

D. $2k^2$

E. None of these



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39. The number of terms in the sequence 3, 6, 12, 24, ... which are between the values 1000 and 100,000 is:

A. 3

B. 7

C. 9

D. 12

E. None of these

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40. The sum of all θ that are solutions to $3\left(\sin^2\theta - \cos^2\theta\right) = 13\sin\theta - 8$ where $-\pi \leq \theta \leq \pi$ is:

A. 2π

B. π

C. 0

D. $\pi/3$

E. None of these

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41. If a polynomial $P(x)$ with integer coefficients is divided by $x^2 + x - 6$, the remainder is $121x - 199$. If the same polynomial $P(x)$ is divided by $x^2 - x - 2$, the remainder is $15x + c$ where c is some integer. The remainder when $P(x)$ is divided by $x^2 + 4x + 3$ is:

A. $136x - 186$

B. $280x + 278$

C. $106x - 214$

D. $327x + 147$

E. $98x - 156$

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42. Pierre Fermat, a French mathematician who lived from 1601-1665, theorized that every prime number of the form $4n + 1$, for $n \geq 1$, is the sum of two squares in one and only one way. For example, 13 can be written in the form $4(3) + 1$ and is equal to $9 + 4$. Now, let A be the largest prime number less than 100 that can be written in the form $4n + 1$. Then, we write $A = p^2 + q^2$. The value of $|p^2 - q^2|$ is:

A. 39

B. 77

C. 13

D. 11

E. 65

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43. If $|x+3| - a = b - |1-x|$ has a solution, then the statement which must be true is:

A. $a+b=0$

B. $a+b \geq 4$

C. $a+b<2$

D. $a-b \geq 0$

E. $a-b<4$

---

44. The remainder when 6^{43} is divided by 11 is equal to:

Hint: Consider the base 2 representation of 43 and use the fact that if $k = ab$, then the remainder when k is divided by n may be computed as follows: find the remainders when a and b are divided by n; compute the product of these remainders; find the remainder when the product is divided by n.

A. 0

B. 3

C. 5

D. 7

E. None of these

---

45. When $x^{100} - x^{99} - x + 1$ is divided by $x^2 - 4x + 3$, the remainder is:

A. 0

B. 3^{99}

C. $(3^{99}-1)x - 1$

D. $(3^{99}-1)x + 1 - 3^{99}$

E. None of these

---

46. If $\sec\theta + \tan\theta = 3$, then we know that $\cot\theta$ is equal to:

A. $\frac{3}{4}$

B. $\frac{4}{3}$

C. $\sqrt{3}$

D. 3

E. $\frac{10}{3}$

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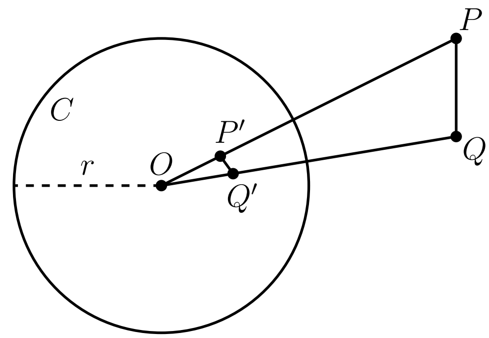
47. The inversion σ across the circle C with radius r is a map defined as follows: for any point R (excluding the origin), $\sigma(R)$ is the point on the ray $\overrightarrow{OR}$ with a distance $r^2/|OR|$ from the origin. A related process, called stereographic projection, has been used to make maps. Here, we have the inverses $P' = \sigma(P)$ and $Q' = \sigma(Q)$. The area $A(\Delta OP'Q')$ is:



A. $\frac{r^2}{|OQ|^2|OP|^2}A(\Delta OQP)$

B. $\frac{r^4}{|OQ|^2|OP|^2}A(\Delta OQP)$

C. $\frac{r^4}{|OQ||OP|}A(\Delta OQP)$

D. $\frac{r^2}{|OQ||OP|}A(\Delta OQP)$

E. None of these

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48. For a fixed natural number $n > 1$, modular arithmetic performed on two integers pertains to remainders upon division by n. In particular, for integers a and b,

$$a \oplus_n b = \text{the remainder when a+b is divided by n and}$$
$$a \odot_n b = \text{the remainder when ab is divided by n.}$$

For example, if $n = 8$, then $3 \oplus 6 = 1$ since $3 + 6 = 9$ and the remainder when 9 is divided by 8 is equal to 1. Similarly, $2 \odot 7 = 6$ since $2 \cdot 7 = 14$ and the remainder when 14 is divided by 8 is equal to 6.

If X is a subset of integers, then the $\odot_n$ multiplicative identity for X is an element e of X for which

$$a \odot_n e = a$$

for all elements a in X. It is possible that a given subset X does not have such an identity element, in which case, the $\odot_n$ multiplicative identity does not exist.

For $n = 10$ and $X = \{2, 4, 6, 8\}$, the $\odot_{10}$ multiplicative identity for X is equal to:

A. 2

B. 4

C. 6

D. 8

E. None of these

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49. Given that $\log_3(p) = \log_4(q) = \log_{\frac{16}{3}}(p+q)$, then the value of $2q/p$ is:

A. $1+\sqrt{2}$

B. $1+\sqrt{3}$

C. 3

D. $1+\sqrt{5}$

E. None of these

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50. Three circles are inscribed in a fourth larger circle. Each circle is mutually tangent to the other three. Let w and x be related as $w = x + 2.5$; also, label the radius of the largest (outer) circle y. The Descartes-Soddy theorem provides one relationship among the variables:

$$2\left(\frac{1}{x^2}+\frac{1}{x^2}+\frac{1}{w^2}+\frac{1}{y^2}\right) = \left(\frac{1}{x}+\frac{1}{x}+\frac{1}{w}+\frac{1}{y}\right)^2$$

If the shaded region's area is 120, the radius x is a root of which 6^{th} degree polynomial?

For compactness, express your answer in terms of $a = x - 4w$, $b = 2w(w+x)$, $c = xw^2$, $d = 2x^2 + w^2$, and $e = \frac{120}{\pi}$.

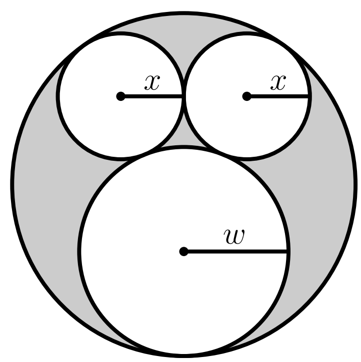
A. $a^2(d+e)^2 + (2ac - b^2)(d+e) + b^2$

B. $b^2(d+e)^2 + (2ac - b^2)(d+e) + c^2$

C. $a^2(d-e)^2 + (2ac + b^2)(d+e) + c^2$

D. $a^2(d+e)^2 + (2ac - b^2)(d+e) + c^2$

E. None of these



Answer Key

1. C
2. E
3. D
4. A
5. C
6. E
7. C
8. A
9. B
10. C
11. E
12. D
13. C
14. D
15. B
16. D
17. C
18. A
19. D
20. E
21. D
22. A
23. A
24. A
25. E
26. A
27. C
28. D
29. D
30. D
31. A
32. B
33. D
34. C
35. D
36. C
37. A
38. E
39. B
40. B
41. B
42. E
43. B
44. D
45. D
46. A
47. B
48. C
49. D
50. D